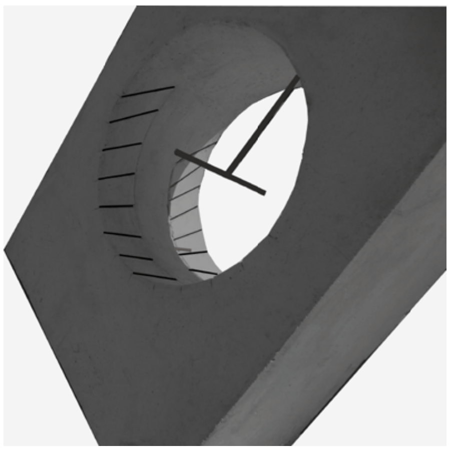
The Enigma of Ai Khanum

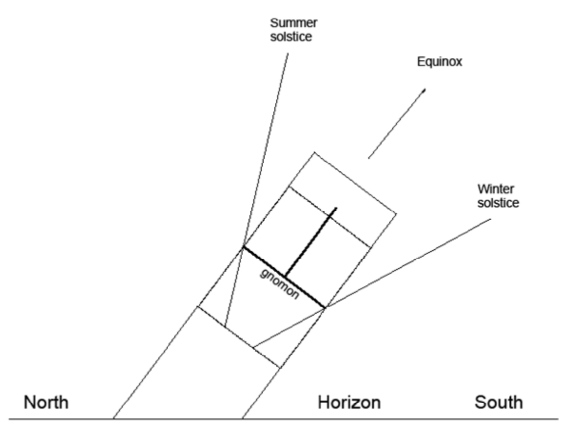
Between 1964 and 1978 a French-led archeological expedition in Afghanistan unearthed a number of artifacts from a Hellenic city in northern Afghanistan. The modern name of the site, Ai Khanoum "lady moon" in Uzbek, was probably Alexandria on the Oxus River. Artifacts from this site are currently (2009) making a museum tour of the United States (Washington, San Francisco, Houston New York). Among many interesting artifacts found in this eastern outpost of Hellenic culture, which was destroyed around 150 BC, were two sundials carved from local limestone. One of them is an unremarkable scaphe



dial, typical of the Greek dials of the period. The second dial, until its recent exposure by the exhibit, was not unknown but languished in the obscurity of the Kabul national museum. It is virtually unique in its design. One would have to look to modern times to find anything similar. In his *De Architectura*, Vitruvius enumerates 12 types of sundial, of which several have remained unknown or speculative. One of these is the *plinthium sive lacunar* dial said to have been invented by Scopus of Syracuse. *Plinthium* is a squared block used for building. *Sive* means "or." *Lacunar* indicates hollowed or emptied out. It is very plausible, although speculative, that this dial represents the only known example of a *plinthium sive lacunar* dial. In any case, there are no other candidates.

The dial in question (illustration) consists of a block of limestone 45 cm by 35 cm by 15 cm thick. It is bored through leaving a cylinder 22 cm in diameter. The gnomon was missing, but traces of it remained in a socket at the top of the cylinder and its inverted T-shape can be inferred. The longer segment of the T is suspended from the socket at the top of the bored cylinder, while the shadow casting element is formed by the inverted T crossbar which coincides with the axis of the cylinder and is equal in length to the thickness of the block. Although the shape of the missing gonomon was inferred by the analysis in the archeological report, it is typical for dials of this period to be fitted with a rod-shaped gnomon whose tip, rather than the shaft, told the hour. The base of the stone block is beveled at an angle of 53°, the co-latitude of Ai Khanoum, situated at 37°, so that the block sits parallel to the equator and polar axis.

The dial has what clearly must be solstice lines inscribed at a distance of 48 mm from each face on the lower half of the cylinder's surface. On each face of the block, the lower half of the cylinder is divided by 13 lines with 15 degree separations marking the hours from sunrise to sunset. With the dial correctly aligned, at the equinox sun's rays would just glance from the tip of the gnomon onto the plane faces of the dial, marking out 12 equal hours from sunrise to sunset. The solstice lines are precisely 48 millimeters from the planar faces of the dial. This distance is where the gnomon tip's shadow will fall at the summer solstice, with sunlight shining on the upper face, and winter solstice, with sunlight shining on the lower face. This distance is a function of the 23.5 constant obliquity of the ecliptic and the radius of the cylindrical bore. The 48 mm solstice lines on the dial



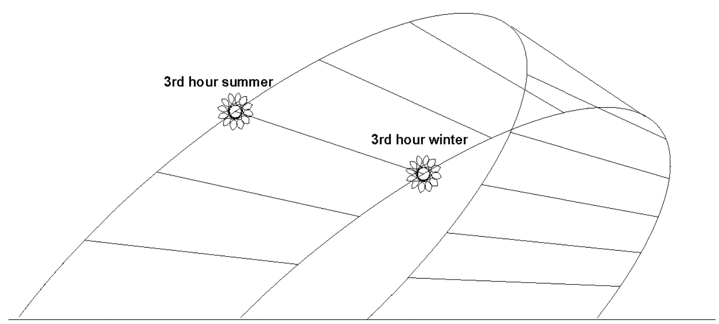
correspond to the 11 cm radius: 48/110 = Tan(23.5).

At first sight, this dial looks strikingly modern, even elegant. The sun's shadow cast by the length of the gnomon will trace out equal hours throughout the year while the gnomon tip will tell the season emerging disappearing from the northern face at the Autumnal equinox and reentering from the southern face until it reaches the winter solstice and reverses course. Not only that, but the dial would be accurate at any latitude, only needing that the base angle be re-cut or wedged to restore its polar-equatorial orientation. This dial is the only known example of its exact type from ancient times, and is quite rare in having a polar-equatorial orientation, a design not prevalent until more than 1000 years later.

But to a casual observer something appears wrong. The hour lines, instead of being perpendicular to the two faces as expected, are cut at an angle. The summer lines splay outward from the dial face toward the interior, while the winter lines angle toward each other. On closer examination it becomes apparent that the hour lines are meant to measure unequal hours: 12 long summer hours and 12 short winter hours. To a modern observer, this is something of a surprise. The alignment and line-casting shape of the gnomon tell equal hours so naturally and clearly that one can easily forget that in ancient times dials were generally made to tell unequal hours, normally pointed by the shadow's tip rather than by a shadow edge coinciding with an inscribed line.

Since time was reckoned in unequal hours for real-life use there was no market, so to speak, for equal-hour dials. The sun rose at 1 (the first hour) and set at 12, (the last hour) all year round. Unequal hours were the real hours, and to be of any practical use, those were the hours that a dial must tell.

The design and functioning of this plinthium dial is easier to grasp if it is seen as variant of the more common scaphe dial. The scaphe dial is essentially a mirror projection of a portion of the celestial sphere (see illustration) onto a matching bowl where the equinox, solstice and hour division lines were indicated by the shadow of a gnomon tip. The plinthium dial is simply a projection of a portion of the sphere onto a corresponding cylinder. The mechanical constraints of producing a useful shadow require that the cylinder be split into summer and winter halves and reoriented so that the equinox lines face outwards.

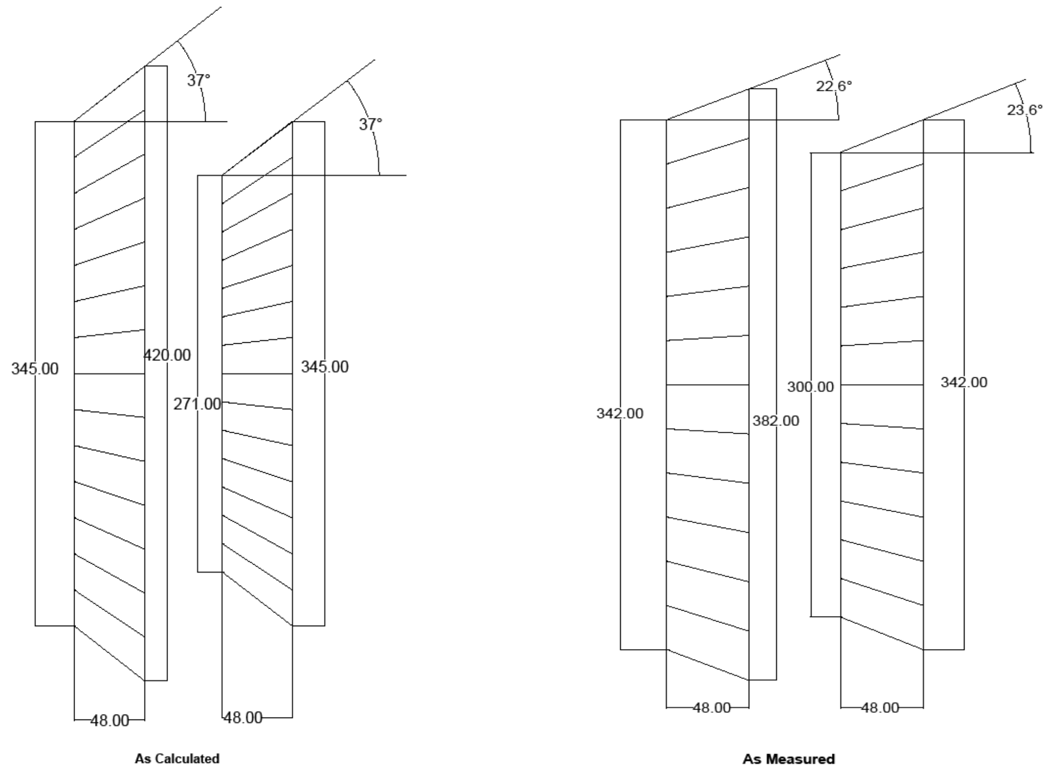


How might the designer of the Ai Khanum dial have derived the layout for the lines inscribed on the surface of the bored cylinder? A modern constructor could do this quite easily. At the equinox, the tip of the gnomon, which is at the exact center of the cylindrical void casts a shadow parallel to the horizon at sunrise and sunset, just grazing the rim on both surfaces, at exactly ½ its circumference. Threfore, the first and last hours (6 AM and 6 PM in modern terms) bisect the rim of the bore at both equinoxes. These arcs simply need to be divided in 12 equal segments to tell time on those two dates. As measured, these equinox arcs – both of which equal half the circumference of the hole, are 345 mm long. The next step requires locating the distance from the two faces of the block to inscribe the arcs corresponding to the two solstices on the inside of the cylinder. This distance, as noted above is a function of the 23.5 obliquity of the obliquity of the ecliptic. It can be found by making a right triangle with a 23.5° acute angle emanating from the gnomon's tip. The adjacent side will be the diameter of the bore, the hypotenuse the distance from the gnomon's tip to the cylindrical surface, and the third, opposite, side of the triangle will be the distance from the surface of the block to the solstice line. The same thing is done for both sides with the triangles facing in opposite directions.

Obtaining the length of the two solstice arcs is more difficult. If the length of the day at summer and winter solstice could be measured empirically, for example with a watch, the lengths of the arcs could be calculated as a proportion between of the 12 hour equinox day, and the corresponding lines inscribed as a proportion of the equinox line. This could theoretically be done by an observer at Ai Khanum, assuming both possession of an independent timepiece and an unobstructed view of sunrise and sunset, but would require a year's wait and two clear days. More likely, a modern constructor would either use a published reference to find day lengths, or more elegantly derive the measurements of the lines using trigonometry, for example by the method cited by Luis Janin in his article on this dial published in 1978[1]: The azimuth of sunrise/sunset on a given date can be calculated with two values: latitude and declination of the sun on the date with the formula cos A = - tan(latitude) * tan(declination). At the solstices the sun's declination will be ± 23.7 so at 37° latitude if we apply these values to the half-diameter of the dial, we obtain a summer arc length of 420 mm and a winter arc length of 271 mm. We already know the offset toward the midpoint of the cylindrical surface from the two faces of the block, so we will only need to measure 420 mm for the length of the summer solstice arc and 211 mm for the length of the winter solstice arc to define them completely. These four arcs, consisting of the two rim half-diameters and the two solstice arcs each just need to be marked off into 12 equal segments, and lines traced between them. This description may be easier to follow by referring to the diagrams below where the lines have been unrolled from the inside of the cylinder, so to speak, and drawn on a flat surface.

To compare the actual hour lines on the dial with the theoretical hour lines corresponding to the 37° latitude, we need accurate measurements taken from the dial itself since it is difficult to obtain them from a photograph. Fortunately such measurements exist and are available from the report of the archeological delegation[2]. They were taken directly by laying moistened paper along the interior face of the dial and tapping it with a stiff brush to pick up the impression from the stone. The paper was removed and laid flat. The lines as measured this way and as calculated theoretically using the method described above, are shown below. The drawing at the left shows the lines as calculated and the drawing at the right shows the lines as measured. While the values are reasonably close for the equinoxes, they diverge substantially for the solstices, and therefore produce very different hour lines.

Arc lengths:	Equinoxes	Summer Solstice	Winter Solstice
Calculated:	345	420	271
Measured	342	382	300



The solstice arcs and hour lines as measured are not correct for 37° where the dial was found, but would be reasonably correct for a dial designed for a latitude of around 23°. Thus, the mystery: how is it possible that a dial so elegantly constructed with an inclination that is virtually exact for the location it was found at, and with inscribed solstice arcs cut at the correct distance from the rim of the cylinder could be so far off in the calculation of the hour lines and the day lengths? And why the 23° latitude? One possibility is that the dial was actually designed for a different location but moved and the base was re-cut for its new home. If you follow the 23 degree latitude line across the globe south of Afghanistan there are very few possibilities. Directly south is the Arabian Sea, to the southwest is the Arabian Desert, the Red Sea and eventually the Nile basin where we find the Ancient Greek city Syene, near modern Aswan, which Eratosthenes famously used in his calculation of the earth's circumference. Toward the Southeast we eventually find the Indian city of Ujjain. This has led to speculation that the dial may have been designed for, or moved from one of those locations. [3]

How plausible is it that the dial was really moved, designed or copied from a different location? It is hard to justify this hypothesis. In the first place, as documented by the archeological report, the dial appears to have been carved from the same type of limestone used for other sculptures found at Ai Khanum. The idea that it was actually made for one of the only two known possible sites at 23 degrees and transported several thousand miles is hard to credit. As evidenced by this and other dials, knowledge of gnomonics was not so arcane as to justify the expense of shipping a block of heavy limestone several thousand stadia and re-cut its angle of inclination.

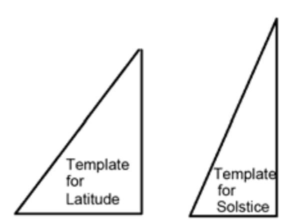
Other hypotheses to explain the error, such as different gnomon alignment, or a different observation of sunrise and sunset due to local terrain, all lead nowhere. It is hard to avoid the conclusion that while the dial was carefully calculated and constructed for its location at Ai Khanum, the hour lines were simply an error. But how could such a mistake have been made? And why 23°, a suspiciously familiar number, close to the obliquity of the ecliptic?

At this point, we enter the realm of speculation, but it is possible to explain the error based on some plausible assumptions about the technical methods available at the time which if correct would also shed light on the methods that might have used for the dial's design and construction.

First of all, the Hellenic gnomonist who designed the dial did not have available the tools we could drew on for the theoretical calculation mentioned above. He obviously did not have a watch to observe the times of sunrise or sunset. Trigonometry must be ruled out. The earliest known use of trigonometry was a table compiled by Hipparchus of Nicea at roughly the same time that Ai Khanum was destroyed. A complex trigonometric calculation such as the one cited above would be out of the question. The unknown gnomonist did not have the luxury of a decimal number system. The units available for linear measurement would only have been standardized locally, and at the time the dial was constructed he probably did not have 360° system for measurement of angles, although he could have used the earlier method based on fractions of a right angle. One assumes that the gnomonist was not the actual stone carver, a profession requiring tools and skills quite different from mathematics. So once the theoretical calculation had been carried out, there would have been the additional problem of transferring all the dimensions correctly to the block of limestone without the sort of standardized objects we are familiar with like millimeter graduated rulers, protractors, vernier calipers and so forth. Therefore, the designer would have had to carry out the calculations somehow, or possibly transcribe them based on instructions in a text, and hand them to the stone carver in some physical form such as drawings or templates so that lines and dimensions could be laid out accurately on the limestone. A reconstructed replica of the dial carried out by the writer suggests that the design itself could have been carried out using simple triangles using straightforward calculations using nothing more abstruse than fractions, and the triangles handed to the stone carver in the form of templates for him to use in laying out the correct measurements. We would use paper or cardboard for this, but papyrus, wood, or thin sheets of lead or bronze could have been used. There are three points in the construction of the dial where an angular measurement would have been needed that could have been supplied by a right-triangle template: 1) the local latitude, measured by what Vitruvius called the "equinoctial shadow" to cut the bottom of the dial 2) the invariant value of the obliquity of the ecliptic to use for both the radius of the bore ("adjacent" side) and the offset of the two solstice arcs from the planes ("opposite" side). 3) an angle to use for setting out the sunrise and sunset lines, which would intersect the solstice arcs and determine their length.

The equinoctial shadow measurement needed to align the dial itself with the equator would have been measured directly using the shadow cast at the equinox and transferred to a template. The second triangular template needed lay out the center and diameter of the cylindrical bore and the offset of the two solstice lines, is simply a right triangle with an acute angle of 23.5 degrees and adjacent side equal to the dimension chosen for the bore. It is not impossible that this was derived using linear and angular units of measurement, but it could have been expressed very easily using a ratio of the opposite and adjacent sides of a right triangle. A convenient ratio approximating the obliquity of the ecliptic is 7/16. A triangle defined by these two numbers is extremely easy to derive (double a value and then remove its eighth part), and having been used to create a physical template is easy to transfer to the physical work. This simple ratio produces triangle with an acute angle of 23.63 degrees which is actually closer to the contemporaneous value of obliquity of the ecliptic in 150 BC than today's value of 23.5.

For the third angle – the one needed to lay out the unequal lines – we have a much simpler approach than the trigonometric calculation given above. Imagine the location of the point on the inside of the cylinder at the moment of sunrise. These points will clearly trace out a line that is parallel to the horizon over the course of the year. This means that the means that the same template used to produce the base-cut latitude could have been used to project the first and last hour lines, which must lie parallel to the base and parallel to the horizon, and in fact only two triangles would be required. One



triangle is used for the diameter of the bore and the offset of the solstices; the other to cut the base to the correct proper latitude and to lay out the sunrise and sunset hour lines parallel to the base. With the first and last hour lines laid out, their intersection with the solstice arcs will define the length of the solstice days. At this point it is only necessary to be divide all four arcs into 12 segments and “connect the dots” to obtain the 12 unequal hour lines.

The erroneous hour lines on this dial could be plausibly explained by the hypothesis that the the stone carver – or perhaps the gnomonist himself – simply used the wrong triangular template. If we now run the calculations backwards using the value of 7/16 in lieu of 23.5 degrees for the template and fine-tune using a one millimeter adjustment of the radius, we obtain the following values, which are almost identical to the as-measured values.

	Equinox arc	Summer arc	Winter arc	Solstice lines
As measured	342	382	300	48
Calculated (110 mm radius)	345.6	387.9	303.3	48.13
Calculated (109 mm radius)	342.4	384.4	300.4	47.69

How would such an obvious mistake go undetected? First of all, there is no reason to assume that it was not. Maybe it was detected and ignored, or maybe this dial was replaced with a corrected one. Second, the mistake would not have been as obvious as it seems. With no clocks available, it was impossible to check the dial against an external fixed standard. The only time when the mistake could have been easily verified against an external event would have been sunrise or sunset, when the shadow would not have fallen on the correct line, but as it happens, the topology surrounding the gymnasium at Ai Khanum leaves the location in shadow at both times. At any other time of the day, the dial would have indicated an incorrect time but could only have been disputed by a simultaneous reading from another dial, such as the scaphe dial found at the same site. If a discrepancy had been noticed one would have needed a third independent reading to break the tie. The true explanation for the enigma of the Ai Khanum dial will never be known for certain, but speculation based on the need for construction techniques to pass the measurements from gnomonic theory to stone carver’s practice leads to back to a hypothesis that the original calculation may have also been carried out using ratios and physical triangles.

[1] Luis Janin, “Un Cadran Solaire Grec a Ai Khanoum, Afghanistan” Bulletin de la Société Astronomique de France. Vol 92, Septembre 1987.

[2] Serge Veuve, Mémoires de la délégation archéologique française en Afghanistan. Tome XXX Fouilles d’Ai Khanoum VI, le Gymnase (Diffusion de Bocard, 1987)

[3] Paul Bernard, “The Greek Colony at Ai Khanum and Hellenism in Central Asia” AFGHANISTAN, Hidden Treasures from the National Museum, Kabul, eds Fredrik Hiebert and Pierre Cambon (National Geographic Society 2008)